

The force density and the kinetic energy-momentum tensor of electromagnetic fields in matter

Rodrigo Medina^{1,*} and J Stephany^{2,†}

¹*Centro de Física, Instituto Venezolano de Investigaciones Científicas, Apartado 20632 Caracas 1020-A, Venezuela.*

²*Departamento de Física, Sección de Fenómenos Ópticos, Universidad Simón Bolívar, Apartado Postal 89000, Caracas 1080-A, Venezuela.*

We determine the invariant expression of the force density that the electromagnetic field exerts on dipolar matter and construct the non-symmetric energy-momentum tensor of the electromagnetic field in matter which is consistent with that force and with Maxwell equations. We recover Minkowski's expression for the momentum density. We use our results to discuss momentum exchange of an electromagnetic wave-packet which falls into a dielectric block. In particular we show that the wave-packet pulls the block when it enters and drags it when it leaves. The usual form of the center of mass motion theorem does not hold for this system but a modified version of the theorem which includes a spin contribution is shown to be satisfied.

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The Abraham-Minkowski controversy on the momentum of the electromagnetic field in matter has a long story. In 1908 Minkowski [1] proposed a non-symmetric energy-momentum tensor. For photons with energy E it implies a momentum nE/c with n the refraction index. A year later Abraham [2], arguing that angular momentum conservation requires the tensor to be symmetric, made a proposal for which photon's momentum is $E/(nc)$. Since then many theoretical and experimental arguments have been exposed which favor one or the other tensor. Reviews of the controversy can be found in [3–6]. Abraham's premise of symmetry was long ago overruled by the discovery of spin, but arguments apparently independent appeared to back his proposal, notably one based in the so called center of mass motion theorem (CMMT), which states that the center of mass of an isolated system moves with constant velocity [7]. The argument says that since Minkowski's momentum in matter is greater than in vacuum, photons crossing a dielectric block will pull the block instead of pushing it and the CMMT will be violated. As we discuss below, the CMMT only holds [8, 9] for systems for which the energy-momentum tensor $T_{\mu\nu}$ is symmetric, that is in the absence of spin. This and other misunderstandings related to the CMMT had populated the literature on the subject with constructions which depart from standard Lorentz-Maxwell electrodynamics. Among them the hidden momentum hypothesis [10, 11] and the use of force densities which are not obtained from the microscopic Lorentz force [3, 4, 12]. In this letter we show that none of this is necessary and that Balzacs argument is wrong. This is done by computing the correct energy momentum tensor of the electromagnetic field in matter and then showing by an explicit computation that for an electromagnetic wave which falls on a dielectric block CMMT does not hold but an improved version of the theorem which includes spin is satisfied.

To discuss the CMMT consider an isolated, localized system with a non-symmetric conserved energy-momentum tensor $\partial_\nu T^{\mu\nu} = 0$ and a non vanishing local spin density $S^{\mu\nu\alpha}$. The total energy $U = \int T^{00} dV$ and the total momentum $p^i = c^{-1} \int T^{i0} dV$ are conserved. The current density of the orbital angular momentum, $L^{\mu\nu\alpha} = x^\mu T^{\nu\alpha} - x^\nu T^{\mu\alpha}$ is not conserved

$$\partial_\alpha L^{\mu\nu\alpha} = T^{\nu\mu} - T^{\mu\nu} . \quad (1)$$

Imposing instead the conservation of the total angular momentum current density $J^{\mu\nu\alpha} = L^{\mu\nu\alpha} + S^{\mu\nu\alpha}$, one has [13, 14],

$$\partial_\alpha S^{\mu\nu\alpha} = T^{\mu\nu} - T^{\nu\mu} . \quad (2)$$

Define the center of mass by

$$X_T^i = \frac{1}{U} \int x^i T^{00} dV . \quad (3)$$

In the case when there is no spin, $T^{\mu\nu}$ is symmetric and the orbital angular momentum $L^{\mu\nu} = c^{-1} \int L^{\mu\nu 0} dV$ is conserved. Then, it is easy to see that the center of mass moves with velocity $c^2 p^i / U$. For the non-symmetric $T^{\mu\nu}$ we are considering it is also easy to see that

$$\dot{X}_T^i = \frac{c}{U} \int T^{0i} dV . \quad (4)$$

In (4) appears the energy current density and not the momentum density. The CMMT is not obtained. This is a consequence of the non-vanishing spin of the system. To see why define the spin matrix $S^{\mu\nu} = c^{-1} \int S^{\mu\nu 0} dV$ and consider the quantity

$$X_S^i = -\frac{c}{U} S^{0i} . \quad (5)$$

From the conservation of the total angular momentum it follows directly that,

$$\begin{aligned} \frac{d}{dt} X_S^i &= \frac{c}{U} \frac{d}{dt} L^{0i} = \frac{1}{U} \frac{d}{dt} \int [x^0 T^{i0} - x^i T^{00}] dv \\ &= \frac{c^2 p^i}{U} - \frac{d}{dt} X_T^i. \end{aligned} \quad (6)$$

The center of mass and spin defined by

$$X_\Theta^i = X_T^i + X_S^i \quad (7)$$

moves with constant velocity $\dot{X}_\Theta^i = c^2 p^i / U$. It is worth noting [15], that X_Θ^i corresponds to the center of mass computed from the symmetric Belinfante-Rosenfeld tensor [16, 17] which is a combination of the energy-momentum tensor and the spin density. In the literature the Belinfante-Rosenfeld tensor is frequently considered as an improved symmetrized energy-momentum tensor but our discussion shows that this interpretation, at least from the mechanical point of view is wrong. Spin and energy-momentum should be distinguished. To illustrate this consider a magnet with total magnetic moment different of zero. The spatial part of the spin density is proportional to magnetization. In the Einstein-de Haas experiment which is used routinely to measure the gyromagnetic ratio [18], spin is converted in orbital angular momentum. This process is described by equation (2) and provides an example where the total energy-momentum tensor clearly cannot be symmetric. For another interesting example see Ref.[8].

We now turn to the computation of the energy-momentum tensor of the electromagnetic field in matter. Contrary to the common belief this can be done unequivocally. The force density on matter is in principle an observable quantity and on theoretical grounds its expression should be deduced from the microscopic Lorentz force. In absence of other interactions the divergence of the energy-momentum of matter is given by this force density. Conservation of momentum then requires that Newton third law holds implying that the divergence of the electromagnetic energy-momentum tensor should be minus the force density. Consequently the key points to solve our problem are to identify the correct density of force which is deduced from the microscopic Lorentz force and to use the action-reaction principle between matter and field. As we show below is also important to take full advantage of the relativistic character of the polarization tensor. So, let us consider a matter system with free charge and current densities ρ and \mathbf{j} , polarization \mathbf{P} and magnetization \mathbf{M} . The bound charge density is $\rho_b = -\nabla \cdot \mathbf{P}$, the bound current density is $\frac{\partial \mathbf{P}}{\partial t}$ and the magnetization current density is $\mathbf{j}_M = c \nabla \times \mathbf{M}$. In the surface of a piece of material there are a surface density of bound charge $\mathbf{P} \cdot \hat{\mathbf{n}}$ and a magnetic surface current density $c \mathbf{M} \times \hat{\mathbf{n}}$. Relativistic invariance is enforced by defining the antisymmetric dipolar density tensor $D_{\alpha\beta}$, with its spatial components obtained

from the magnetization density by $D_{ij} = \epsilon_{ijk} M_k$ and its temporal components given by the electric polarization, $D_{0k} = -D_{k0} = P_k$. The charges and currents associated with \mathbf{P} and \mathbf{M} are encoded in the dipolar four current $j_{\text{dip}}^\mu = c \partial_\nu D^{\mu\nu}$, which like the free charge four current j^μ , is conserved: $\partial_\mu \partial_\nu D^{\mu\nu} = 0$. We work in Gauss units, the metric tensor is $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and c is the speed of light in vacuum. Maxwell equations are

$$\partial_\nu F^{\mu\nu} = \frac{4\pi}{c} (j^\mu + j_{\text{dip}}^\mu), \quad (8)$$

where $F^{\mu\nu}$ is the electromagnetic field tensor. Defining the tensor of magnetizing field \mathbf{H} and electric displacement \mathbf{D} through $H^{\mu\nu} = F^{\mu\nu} - 4\pi D^{\mu\nu}$, the field equations become $\partial_\nu H^{\mu\nu} = 4\pi c^{-1} j^\mu$.

Let us first consider briefly the case with vanishing \mathbf{P} and \mathbf{M} . In this case Maxwell's equations read $\partial_\nu F^{\mu\nu} = 4\pi c^{-1} j^\mu$. The force density on the free charges is a four vector given by $f_{\text{ch}}^\mu = \frac{1}{c} F_{\nu}^\mu j^\nu$. Consider now the gauge invariant symmetric tensor

$$T_S^{\mu\nu} = -\frac{1}{16\pi} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} + \frac{1}{4\pi} F_{\alpha}^\mu F^{\nu\alpha}. \quad (9)$$

The relation

$$\partial_\nu T_S^{\mu\nu} = -\frac{1}{c} F_{\nu}^\mu j^\nu = -f_{\text{ch}}^\mu \quad (10)$$

is an identity which holds for every solution of Maxwell equations. One is allowed to identify $T_S^{\mu\nu}$ as the energy-momentum tensor of the electromagnetic field and to interpret the right hand side of (10) as the force the matter exerts on the field. In particular Newton's action-reaction law holds.

Consider now the case with non-vanishing $D^{\mu\nu}$. Although some authors suppose that the force on matter is of the form f_{ch}^μ with j^μ substituted by $j^\mu + j_{\text{dip}}^\mu$ (See for example [19, 20]) it is easy to be convinced that this is not the case. We obtain the force density expression assuming that 1) The total force on a piece of material is the sum of the forces on each element of the piece. 2) The force on an element equals the force on the dipoles $d\mathbf{m} = \mathbf{M}dV$ and $d\mathbf{d} = \mathbf{P}dV$. The force on a magnetic dipole \mathbf{m} is known to be [21] $\mathbf{F}_{\text{dip}} = \nabla(\mathbf{B} \cdot \mathbf{m})$. The power transferred to matter is $\frac{dW}{dt} = -\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{m}$. Using that in this case $\mathbf{P} = 0$, the relativistic force density on the microscopic dipoles is $f_{\text{dip}}^\mu = 2^{-1} D_{\alpha\beta} \partial^\mu F^{\alpha\beta}$. For non vanishing \mathbf{P} and \mathbf{M} , by relativistic invariance the total force density four-vector is

$$f^\mu = f_{\text{ch}}^\mu + f_{\text{dip}}^\mu = \frac{1}{c} F^{\mu\nu} j_\nu + \frac{1}{2} D_{\alpha\beta} \partial^\mu F^{\alpha\beta}. \quad (11)$$

A related expression is discussed in [24]. The energy-momentum tensor of matter satisfies,

$$\partial_\nu T_{\text{matter}}^{\mu\nu} = f^\mu. \quad (12)$$

Equation (10) is an identity which follows from Maxwell's equations. Using (8) in this case we can write directly the new identity

$$\partial_\nu T_S^{\mu\nu} = -\frac{1}{c} F_\nu^\mu (j^\nu + j_{\text{dip}}^\nu) = -\frac{1}{c} F_\nu^\mu (j^\nu + c \partial_\alpha D^{\nu\alpha}) \quad (13)$$

The right hand side of (13) is not minus the total force on the matter (11) and the identification of $T_S^{\mu\nu}$ as the energy-momentum of the field does not hold. One regains a clear physical interpretation by defining

$$T_{\text{FK}}^{\mu\nu} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} \eta^{\mu\nu} + \frac{1}{4\pi} F_\alpha^\mu H^{\nu\alpha} . \quad (14)$$

which after a simple manipulation using Bianchi's identity, Eq. (13) is shown to satisfy

$$\partial_\nu T_{\text{FK}}^{\mu\nu} = -f^\mu . \quad (15)$$

Newton's third law between matter and field is recovered if one identify $T_{\text{FK}}^{\mu\nu}$ as the kinetic energy-momentum tensor of the electromagnetic field. Of course different energy-momentum tensors may be used for particular purposes, but $T_{\text{FK}}^{\mu\nu}$ is the one that should be used to discuss exchange of linear momentum between matter and the electromagnetic field because it is Newton's third law which guarantees the conservation of the total energy-momentum tensor. With this tensor the energy density is

$$u = T_{\text{FK}}^{00} = \frac{1}{8\pi} (E^2 + B^2) + \mathbf{E} \cdot \mathbf{P} , \quad (16)$$

and Poynting vector and the momentum density are

$$\mathbf{S} = c T_{\text{FK}}^{0i} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} , \quad \mathbf{g} = c^{-1} T_{\text{FK}}^{i0} = \frac{1}{4\pi c} \mathbf{D} \times \mathbf{B} . \quad (17)$$

Maxwell's stress tensor is

$$T_{\text{FK}}^{ij} = \frac{1}{8\pi} (E^2 + B^2) \delta_{ij} - \mathbf{B} \cdot \mathbf{M} \delta_{ij} - \frac{1}{4\pi} (E_i D_j + H_i B_j) . \quad (18)$$

The obtained tensor is different to Minkowski's and Abraham's tensors. Minkowski's tensor in our notation reduces to

$$T_{\text{Min}}^{\mu\nu} = T_{\text{FK}}^{\mu\nu} + \frac{1}{4} F^{\alpha\beta} D_{\alpha\beta} \eta^{\mu\nu} . \quad (19)$$

It differs from T_{FK} by diagonal terms. Poynting's vector and the momentum density are the same for both tensors but the classical Minkowski or Poynting energy density $u_{\text{Min}} = (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})/8\pi$ [22] is different from the expression (16). The diagonal terms of the Maxwell tensor are also different. Abraham's tensor cannot be written in covariant form. This fact was shown in Ref.[23] by an explicit computation and has also a simple demonstration because there is a unique four-tensor that has some particular temporal row in every reference frame and Abraham's and Minkowski's two indices objects have the

same temporal row. Our tensor is related but different to the one obtained by de Groot and Suttorp in a particular case [24]

The non-symmetric part of $T_{\text{FK}}^{\mu\nu}$ has to be interpreted in view of equation (2) as a dipolar torque density

$$\tau_{\text{dip}}^{\mu\nu} = D^{\mu\beta} F_\beta^\nu - D^{\nu\beta} F_\beta^\mu \quad (20)$$

Inspecting its components one observes that indeed the spatial part is given by

$$(\tau_{\text{dip}})_k = \frac{1}{2} \epsilon_{ijk} \tau_{\text{dip}}^{ij} = (\mathbf{P} \times \mathbf{E} + \mathbf{M} \times \mathbf{B})_k , \quad (21)$$

which is the expected torque that the field should exert on magnetic and electric dipoles. The temporal part is,

$$\tau_{\text{dip}}^{0k} = (-\mathbf{P} \times \mathbf{B} + \mathbf{M} \times \mathbf{E})^k . \quad (22)$$

and as we discuss in the following example plays an important role in disentangling the paradoxes of Balazs construction.

The best test for the energy-momentum tensor and the force density presented in this letter is to compute the momentum and energy exchange between a packet of electromagnetic waves and a dielectric medium. Suppose that the region $x > 0$ is filled by a non-dispersive material with dielectric constant ϵ and magnetic permeability μ . A packet of linearly polarized plane waves approaches the yz surface traveling in the x direction. Its electric field is

$$\mathbf{E}_1(x, y, z, t) = E_1 g(t - x/c) \theta(-x) \hat{y} . \quad (23)$$

E_1 is an amplitude, θ is the Heaviside step function and $g(t)$ is a dimension-less well-behaved but otherwise arbitrary function that vanishes for $t < 0$ and $t > T$. At the surface of the material $x = 0$ the packet is reflected and transmitted. The reflected and transmitted packets are

$$\mathbf{E}_2(x, y, z, t) = E_2 g(t + x/c) \theta(-x) \hat{y} , \quad (24)$$

$$\mathbf{E}_3(x, y, z, t) = E_3 g(t - x/v) \theta(x) \hat{y} , \quad (25)$$

where the speed of light in the material is $v = c/n$ with $n = \sqrt{\epsilon\mu}$. For $t < 0$ only the incident packet is present, for $t > T$ the reflected one is in $x < 0$ and the transmitted one is in $x > 0$. For $0 < t < T$ the three packets are touching the surface $x = 0$. The corresponding magnetic fields of the three packets are

$$\mathbf{B}_1 = B_1 g(t - x/c) \theta(-x) \hat{z} , \quad (26)$$

$$\mathbf{B}_2 = B_2 g(t + x/c) \theta(-x) \hat{z} , \quad (27)$$

$$\mathbf{B}_3 = B_3 g(t - x/v) \theta(x) \hat{z} . \quad (28)$$

Using Maxwell's equations the magnetic amplitudes are

$$B_1 = E_1 , \quad B_2 = -E_2 , \quad B_3 = \sqrt{\epsilon\mu} E_3 . \quad (29)$$

By the continuity conditions at $x = 0$

$$E_2 = \frac{1 - \sqrt{\epsilon/\mu}}{1 + \sqrt{\epsilon/\mu}} E_1 , \quad E_3 = \frac{2}{1 + \sqrt{\epsilon/\mu}} E_1 . \quad (30)$$

For $t < 0$ the energy of a cylindrical piece of the incident packet with axis parallel to x and cross section A is,

$$U_1 = \int T_S^{00}(1) dV = \frac{Ac\bar{T}}{4\pi} E_1^2 \quad (31)$$

with

$$\bar{T} = \int_0^T g(t)^2 dt . \quad (32)$$

The momentum of the incident wave-packet is

$$\mathbf{p}_1 = \int \mathbf{g}(1) dV = \int c^{-1} T_S^{i0}(1) \hat{\mathbf{e}}_i dV = \frac{U_1}{c} \hat{x} . \quad (33)$$

For the reflected packet ($t > T$) the energy and momentum are

$$U_2 = \frac{Ac\bar{T}}{4\pi} E_2^2 , \quad \mathbf{p}_2 = \int \mathbf{g}(2) dV = -\frac{U_2}{c} \hat{x} . \quad (34)$$

The energy and momentum transferred to the $x > 0$ side of the space are

$$U_1 - U_2 = \frac{Ac\bar{T}}{4\pi} (E_1^2 - E_2^2) = \frac{Ac\bar{T}}{4\pi} E_3^2 \sqrt{\epsilon/\mu} \quad (35)$$

$$\mathbf{p}_1 - \mathbf{p}_2 = \frac{A\bar{T}}{4\pi} (E_1^2 + E_2^2) \hat{x} = \frac{A\bar{T}}{8\pi} E_3^2 (1 + \epsilon/\mu) \hat{x} . \quad (36)$$

The EM energy and momentum of the transmitted packet are

$$U_3 = \int T_{\text{FK}}^{00}(3) dV = \frac{Ac\bar{T}}{8\pi\sqrt{\epsilon\mu}} E_3^2 (\epsilon\mu + 2\epsilon - 1) , \quad (37)$$

$$\mathbf{p}_3 = \int \mathbf{g}(3) dV = \frac{A\bar{T}v}{4\pi c} E_3^2 \epsilon \sqrt{\epsilon\mu} \hat{x} . \quad (38)$$

Using (11) the power on the matter at time t is obtained

$$\begin{aligned} \dot{W} &= c \int f^0 dv = - \int (\mathbf{P} \cdot \dot{\mathbf{E}} + \mathbf{M} \cdot \dot{\mathbf{B}}) dV \\ &= - \frac{Ac}{8\pi\sqrt{\epsilon\mu}} E_3^2 (\epsilon\mu - 1) g(t)^2 . \end{aligned} \quad (39)$$

Integrating the time the work done on matter is

$$W = - \frac{Ac\bar{T}}{8\pi\sqrt{\epsilon\mu}} E_3^2 (\epsilon\mu - 1) . \quad (40)$$

This work changes the energy of the matter where the wave-packet is located, so it has to be added to the EM energy in order to obtain the total transmitted energy $U'_3 = U_3 + W$. Energy conservation is satisfied $U'_3 = U_1 - U_2$. It is easy to see that U'_3 is the energy of the transmitted packet computed with u_{Min} . Note also that $\mathbf{p}_3 = c^{-1} U'_3 n \hat{x}$ as would be expected for Minkowski's momentum.

To verify momentum conservation one has to compute the impulse on matter. The force on matter has a volume

component given by (11) and a surface component due to the discontinuity at $x = 0$. The volume component is

$$\begin{aligned} \mathbf{F}_V &= \int (P_i \nabla E_i + M_i \nabla B_i) dV \\ &= \frac{A}{8\pi} \int_0^\infty [(\epsilon - 1) \partial_x E^2 + (1 - 1/\mu) \partial_x B^2] dx \hat{x} \\ &= - \frac{AE_3^2}{8\pi} (\epsilon\mu - 1) g(t)^2 \hat{x} . \end{aligned} \quad (41)$$

The surface component of the force at $x = 0$ is equal to the momentum flux exiting the vacuum side minus the momentum flux entering the matter side. That is

$$\mathbf{F}_S = A(T_S^{11}(-) - T_{\text{FK}}^{11}(+)) \hat{x} . \quad (42)$$

Using (9) and (14)

$$T_S^{11}(-) - T_{\text{FK}}^{11}(+) = \frac{\epsilon E_3^2}{8\pi} (1/\mu + \mu - 2) g(t)^2 . \quad (43)$$

Therefore the total force is

$$\mathbf{F} = \mathbf{F}_V + \mathbf{F}_S = \frac{AE_3^2}{8\pi} (1 + \epsilon/\mu - 2\epsilon) g(t)^2 \hat{x} . \quad (44)$$

We note that if diamagnetism does not prevail the wave packet pulls the dielectric. The impulse is

$$\mathbf{I} = \int \mathbf{F} dt = \frac{A\bar{T}E_3^2}{8\pi} (1 + \epsilon/\mu - 2\epsilon) \hat{x} . \quad (45)$$

The total momentum transferred to $x > 0$ for $t > T$ is

$$\mathbf{I} + \mathbf{p}_3 = \frac{A\bar{T}E_3^2}{8\pi} (1 + \epsilon/\mu) \hat{x} = \mathbf{p}_1 - \mathbf{p}_2 \quad (46)$$

as it should be.

Let us turn to the motion of the center of mass of the system. It is convenient to separate the electromagnetic and matter contributions to the center of mass and write

$$\mathbf{X}_T(t) = \mathbf{X}_{\text{FT}}(t) + \mathbf{X}_{\text{MT}}(t) \quad (47)$$

When the wave is moving towards the dielectric there is no spin contribution to the center of mass and spin and we may write

$$\dot{\mathbf{X}}_\Theta = \dot{\mathbf{X}}_T = \dot{\mathbf{X}}_{\text{FT}} = \frac{c^2 p_1}{U_1} \hat{x} , \quad t < 0 . \quad (48)$$

The position of the center of mass of the transmitted wave-packet for $t > T$ is

$$\mathbf{X}_{\text{FT}}(t) = \frac{1}{U_3} \int x u dV \hat{x} = \frac{1}{vT} \int x g(t - x/v)^2 dx \hat{x} . \quad (49)$$

It immediately follows that $\mathbf{X}_{\text{FT}}(t) = \mathbf{X}_{\text{FT}}(0) + tv\hat{x}$. The center of mass velocity of this packet $\dot{\mathbf{X}}_{\text{FT}} = v\hat{x}$ is in this case indeed constant and can be easily expressed as

$$\dot{\mathbf{X}}_{\text{FT}} = \frac{1}{U'_3} \int \mathbf{S} dV = \frac{1}{U'_3} \int T_S^{oi}(1) \hat{\mathbf{e}}_i dV , \quad (50)$$

but the strong CMMT does not hold ($\mathbf{p}_3 \neq c^{-2}U'_3\dot{\mathbf{X}}$) since $\mathbf{g} \neq c^{-2}\mathbf{S}$. If the momentum of the transmitted wave-packet were Abraham's the CMMT would be satisfied but the momentum conservation would be lost. Let us then compute the spin contribution. After the wave has penetrated the dielectric, the center of mass of matter satisfies Newton's second law $m\dot{\mathbf{X}}_{\text{MT}} = \mathbf{I}$ where m is the mass of the dielectric block and \mathbf{I} is the impulse computed in (45). The spin density has contributions from matter and field and satisfies equation (2). The separation of these contributions is an difficult and interesting problem which is not necessary to discuss here. Since the matter contribution to the energy-momentum tensor is symmetric, using (20) we have

$$\partial_\alpha S^{\mu\nu\alpha} = \tau_{\text{dip}}^{\mu\nu}. \quad (51)$$

with $\tau_{\text{dip}}^{\mu\nu}$ given by (21) and (22). Focusing in the temporal components which are the ones that contribute to (5) we have,

$$\partial_\alpha S^{0k\alpha} = (-\mathbf{P} \times \mathbf{B} + \mathbf{M} \times \mathbf{E})^k = -\frac{\mu\epsilon - 1}{4\pi\mu}(\mathbf{E} \times \mathbf{B})^k$$

where we use the constitutive equations $4\pi\mathbf{P} = (\epsilon - 1)\mathbf{E}$, $4\pi\mu\mathbf{M} = (\mu - 1)\mathbf{B}$. Now, spin transport in this system is due by the drift, $S^{0ki} = S^{0k0}v_m^i$ with v_m^i the matter velocity which in this case vanishes. Then $\partial_i S^{0ki} = 0$ and using that the right hand side of (52) points in the x direction we have

$$\partial_0 S^{010} = -\frac{(\mu\epsilon - 1)\sqrt{\epsilon\mu}}{4\pi\mu}g^2(t - x/v)E^2 \quad (52)$$

Integrating in space the spin term which appear in equation (6) is for $t > T$

$$\frac{\partial}{\partial t} S^{010} = -\frac{AcE^2\bar{T}(\mu\epsilon - 1)}{4\pi\mu} = -\frac{U_1\dot{X}_S^1}{c}. \quad (53)$$

Taking all together, for $t > T$ we verify that for $t > T$,

$$\begin{aligned} \dot{X}_\Theta^1 &= -\frac{U_2c + (U_3 + W)v + I}{U_1c^2} + \frac{\dot{X}_S^1}{U_1c^2} \\ &= \frac{A\bar{T}E_1^2}{4\pi} = \frac{c^2p_1}{U_1} \end{aligned} \quad (54)$$

as requested by the improved theorem (6).

CONCLUSION

Using relativistic invariance and Maxwell equations we deduce an invariant expression of the force density that the electromagnetic field exerts on dipolar matter (11). Imposing Newton's third law between the field and matter, we construct the kinetic energy-momentum tensor of the electromagnetic field in matter $T_{\text{FK}}^{\mu\nu}$. Our result differs from both Minkowski and Abraham proposals but

settles the Minkowski-Abraham controversy about the momentum density in favor of the former. The energy density obtained is not Poynting's classical expression but energy conservation is assured by the power contribution of the dipolar term in Eq.(11).

We use force density and $T_{\text{FK}}^{\mu\nu}$ to verify energy and momentum conservation in the interaction of a packet of electromagnetic waves with a dielectric medium. We show that in this system the CMMT does not hold but the modified equation (6) is satisfied with a non trivial contribution of the temporal spin.

We have shown, in opposition to the argument of Balazs [7], that for $n > 1$ the wave packet pulls the material when it enters a medium (See Eq.(44)). Experimental support to this result was reported in [25]. Since there has been some perplexity about this possibility, we note that it has a very simple physical explanation. Dielectric and paramagnetic materials are attracted while diamagnetic materials are repelled in the direction to high field regions, so when the wave packet is entering the medium it pulls the material unless diamagnetism prevails. For the same reason when the wave leaves, it drags the block.

In general Minkowski's tensor is not particularly useful but for a material with non-dispersive linear polarizabilities ($D_{\alpha\beta} = \chi_{\alpha\beta\mu\nu}F^{\mu\nu}$), such as the one discussed above, it may be interpreted as the energy-momentum tensor of the electromagnetic field plus the fraction of the energy of the matter that is due to the polarizations. Nevertheless its divergence is not the reaction of the force acting on the matter.

We also want to mention that the expression for $T_{\text{FK}}^{\mu\nu}$ may also be obtained starting from the microscopic equations and using an averaging procedure[26] or using Noether's theorem within the Lagrangian formalism [14].

* rmedina@ivic.gob.ve

† stephany@usb.ve

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